ACCURATELY ESTIMATING AND BUILDING THE YIELD CURVE

RISK Yield Curve Course, October 1999

Thomas S. Coleman

tcoleman@closemountain.com

ÆQUILIBRIUM INVESTMENTS LIMITED

Outline

- General Approach to Fitting Yield Curve
- Mathematics of Yield & Forward Curve
- Simple Example
- Use of and Criteria for Curves
- Choice of Input Data
- Various Functional Forms

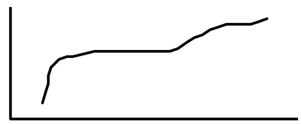
Fitting The Yield Curve - Outline

- General Approach
  - Define discount function with a functional form for forward curve
  - Choosing market data (inputs) and appropriately describing the instruments
  - Define and implementing an appropriate objective function and fitting methodology
  - All instruments priced through discount function

Fitting the Yield Curve - Diagram

Discount Function

Fwd Curve

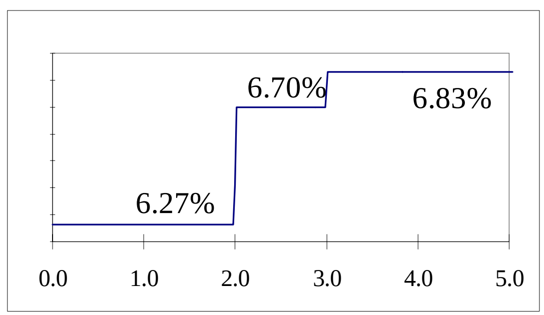


Market Data & Instruments

Type	Matur	Freq	Rate
Swap	2yr	Semi	6.36
Swap	3yr	Semi	6.50
Swap	5yr	Semi	6.66

Objective Function and Fitting Method

Swaps



Options

Fitting - Implementation Issues

- Modularize
  - Separate curve from instrument details
- Re-use code
  - Use DiscFact in curve and swap pricing
  - Use same subroutines to price instruments
- Build in flexibility
  - Changing forward curve and instruments

Yield Curve Mathematics

- Term structure of interest rates expressed as
  - forward curve
  - zero curve
  - discount curve
- I like to use forward curve, but matter of taste

Yield Curve Mathematics - cont'd

- Discount curve in terms of zeros / forwards

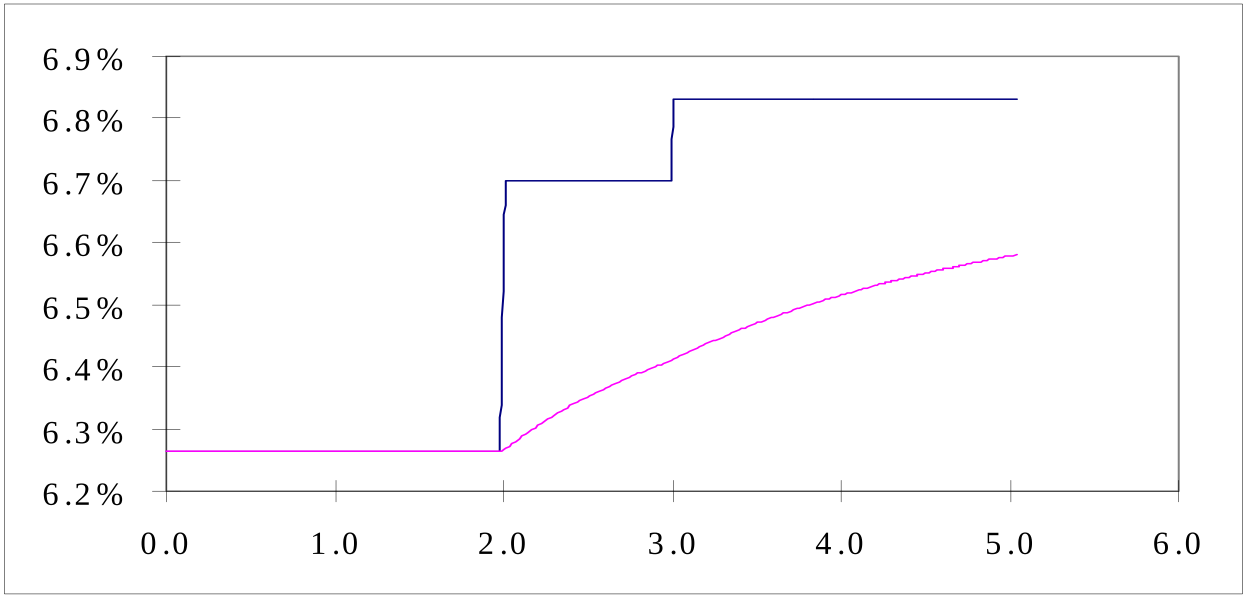
$$df(t) = e^{[-y(t)\cdot t]} \qquad df(t) = \exp\left[-\int_0^t f(u)du\right]$$

- Relation between forwards and zeros:

$$y(t) = \left[\int_0^t f(u)du\right] / t$$

Yield Curve Mathematics - cont'd

- These expressions are for continuously compounded forward and zero rates
- Zero is an "average" of forwards - smoothed



Example - Forward Curve

- Forward curve functional form
  - Piece-wise constant forwards f_1, f_2, f_3
  - Breaks at 2, 3, 5 years
- Discount factor function

$df(t) = \exp[-f_1 * t]$	for $t <= 2$
$df(t) = \exp[-2f_1 - f_2*(t-2)]$	for $2 < t <= 3$
$df(t) = \exp[-2f_1 - f_2 - f_3*(t-3)]$	for $3 < t <= 5$

Example - Market data

- Just swaps for example

Type	Matur	Freq	Rate
Swap	2 yrs	Semi	6.36
Swap	3 yrs	Semi	6.50
Swap	5yrs	Semi	6.66

- NPV of swap

$$NPV = \sum_{i=1}^{2*yrs} df(i/2) \cdot rate/2 + 100 \cdot df(yrs) - 100$$

Example - Objective Function

- Fit forwards so that all NPVs are zero

  $NPV(2yr; f_1, f_2, f_3) = 0$

  $NPV(3yr; f_1, f_2, f_3) = 0$

  $NPV(5yr; f_1, f_2, f_3) = 0$

- Fit forwards sequentially (bootstrap)
  - Fit f_1 by solving $NPV(2yr; f_1) = 0$ since 2 year swap depends only on first forward rate.
  - Then fit f_2 by solving $NPV(3yr; f_1, f_2) = 0$

Example - Results

- First forward is 2 yr rate

  6.26%cc = 6.36%sab

- Second and third forwards easy to bootstrap
  - Can do it with HP12C

  f_2 6.70%cc = 6.81%sab

  f_3 6.83%cc = 6.94%sab

Example - Extensions

- Extending this is easy conceptually
- As always, devil is in the details
  - Swap payments may not fall on exact dates - e.g. holidays
  - Futures and deposits
- This general approach - separating forward curve from instrument - simplifies details
  - Details encapsulated in instrument subroutine

Curve Criteria

- Mark-to-market (interpolator)
  - Curve used for daily MTM of derivatives portfolio
  - E.g. swaps portfolio from liquid futures & swaps
- Rich-cheap analysis (smoothing)
  - Curve used to identify instruments whose market price is rich or cheap relative to others
  - E.g. US Treasury curve with 200 bonds

Curve Criteria - MTM

- Relatively few inputs, each reprices exactly
- Speed and simplicity
- Localization
- Reasonably smooth forwards

Curve Criteria - Rich/Cheap

- Generally many inputs, none fit exactly
- Smoothing noisy data to reasonable market curve
- Strong localization not required
- Speed and simplicity less important

Choice of Input Data

- Depends largely on use of the curve
- For MTM
  - Liquid instruments with good, easily observed market quotes
  - Instruments actually used to hedge the book

Input Data - Swaps MTM

- Generally three “sectors”
  – Money market - libor deposits
  – FRA / Futures
  – Swaps

Input Data - Deposits

- Generally needed to “tie-down” the front of the curve
  - In US I would use over night, 1 week, 2 week, 1 month, then switch to futures
  - Exact deposits used depends on futures dates
- But beware of liquidity problems with longer (e.g. 6 month) deposits
  - Longer deposits and shorter futures may not always match - choose liquidity?

Input Data - FRA / Futures

- Choose between FRAs and futures based on liquidity and transparency
  - In USD, CAD, GBP, EUR I would use futures. In some other currencies FRAs
- Big issue of convexity
  - FRA payoff is convex in rate
  - Futures payoff is linear in rate ($25 / tick)

Input Data - Futures Convexity

- Deposits, FRAs, swaps all same "class" of instrument
  - Can synthetically construct one from the other
  - Arbitrage
- Futures - with linear payoff - is different
- Futures is not simply PV off FRA curve
  - Must use term-structure model to price
  - There are approximations for the convexity correction

Input Data - Convexity Approximation

- Doust’s approximation for convexity correction:

$R_{fut} = R_{fra} / DF_{exp}$^[½$\sigma^2$t*(t+½)/(t+¼)]

t = time to futures expiry (in years)

R_{fra} = forward (FRA) rate from the curve

DF_{exp} = discount rate to futures expiry date

σ = volatility in decimal, i.e. 0.20.

P. Doust, “Relative Pricing Techniques in the Swaps and Options Markets,” *J. Financial Engineering*, March 1995

Input Data - Swaps

- Instruments straightforward
  - But must get frequency, day-count, etc., correct
  - Details change between markets - advantage of separating curve and instruments
- Where to switch from futures to swaps
  - Depends on liquidity and hedge instruments
  - We used 4 years of futures, 5 year swap

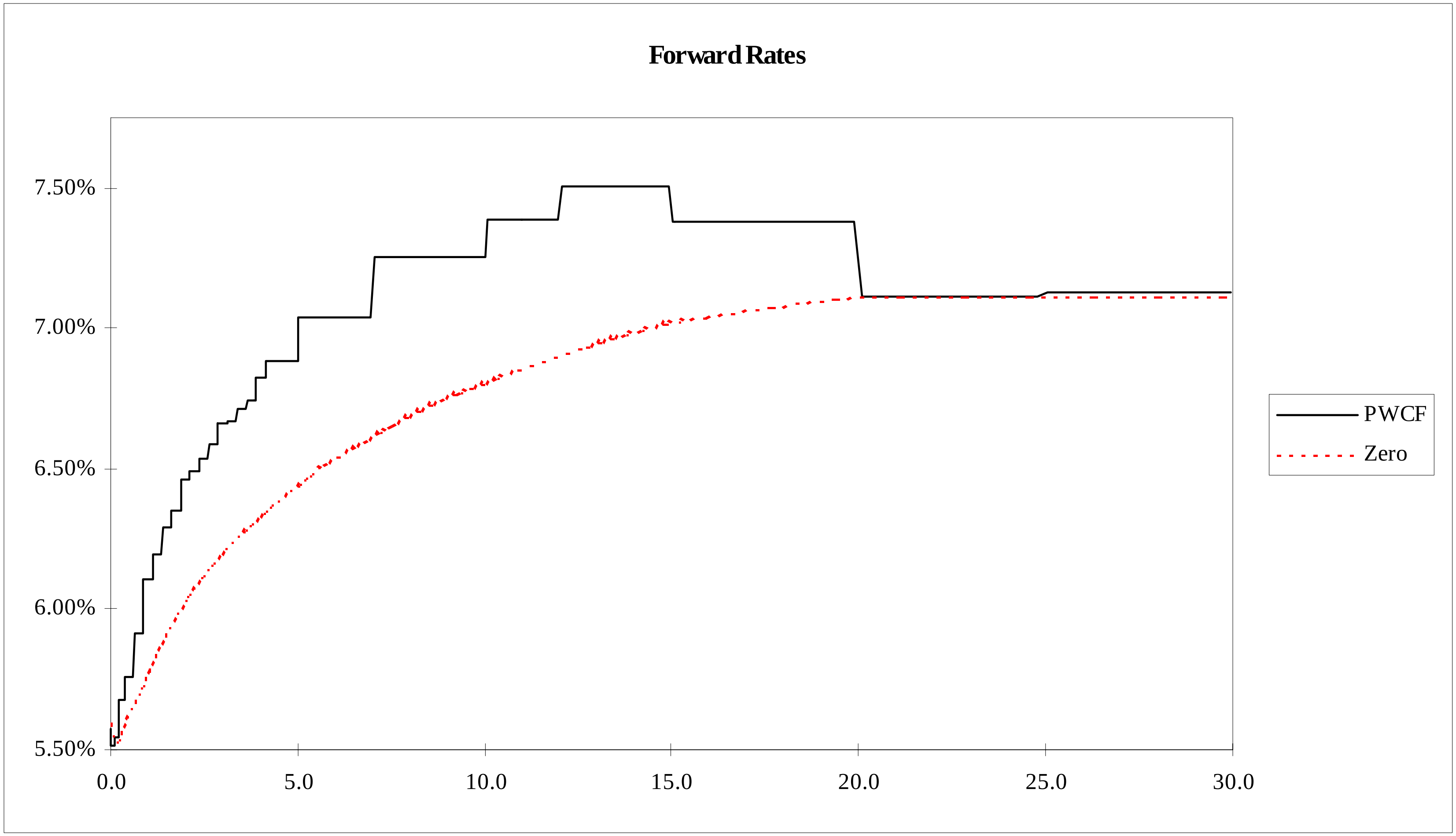
Functional Forms

- Discuss three (four)
  - Piece-wise constant forward (PWCF)
  - Piece-wise linear zeros (PWLZ)
  - Piece-wise linear forwards (PWLF) - twisted and smoothed
- Do not discuss cubic splines
  - Popular, but problems with non-localization
  - In my opinion, not a good form for MTM

Functional Forms - PWCF

- Choose break points (usually instrument maturities)
- Forwards constant between breaks

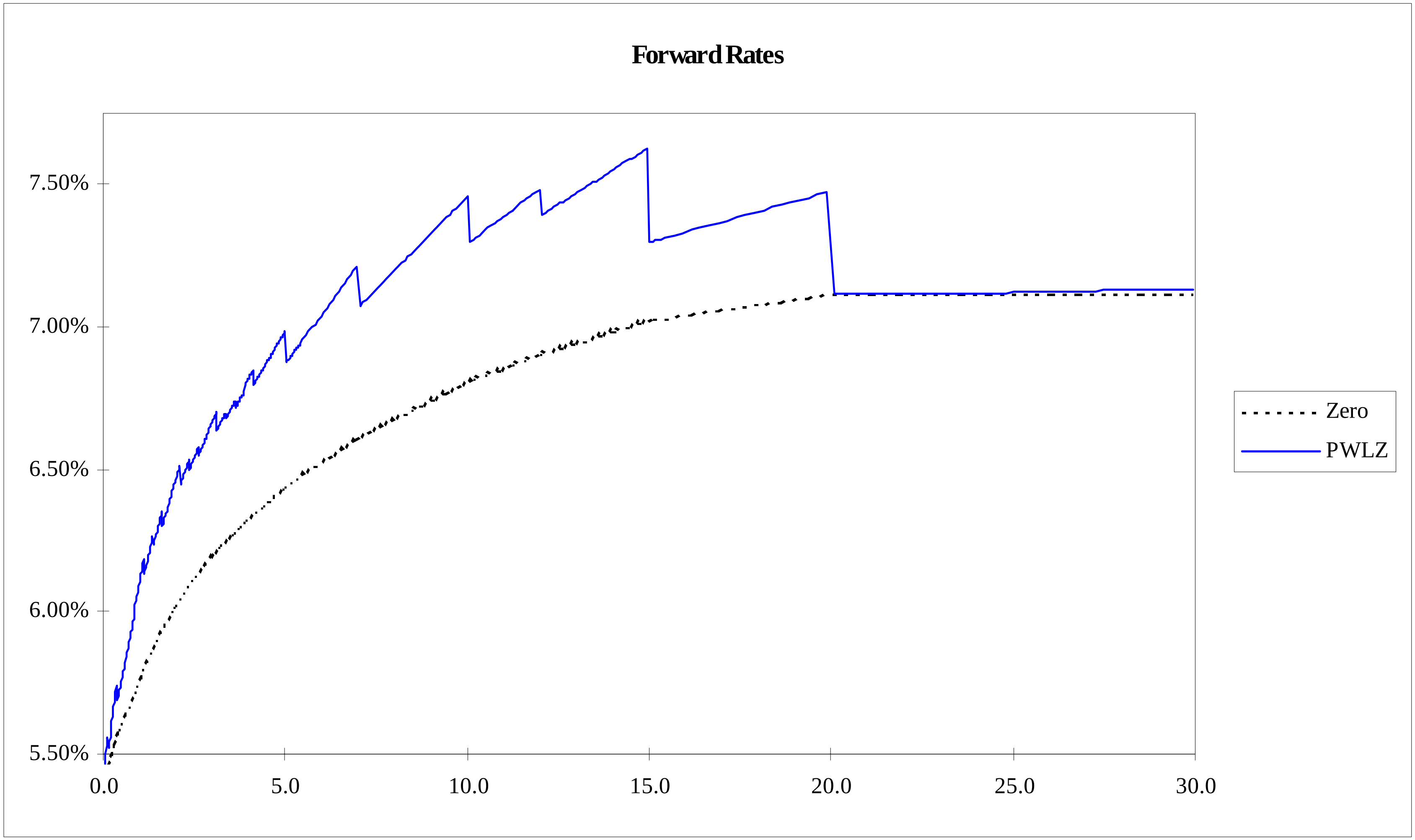
Functional Forms - PWCF



Functional Forms - PWLZ

- Choose break points (usually instrument maturities)
- Zeros linear between breaks
  - Zeros linear, continuous across breaks (knots) but not smooth
  - Forwards linear between breaks, discontinuous across breaks
- Most common market method (or close)
- Large jumps in forwards

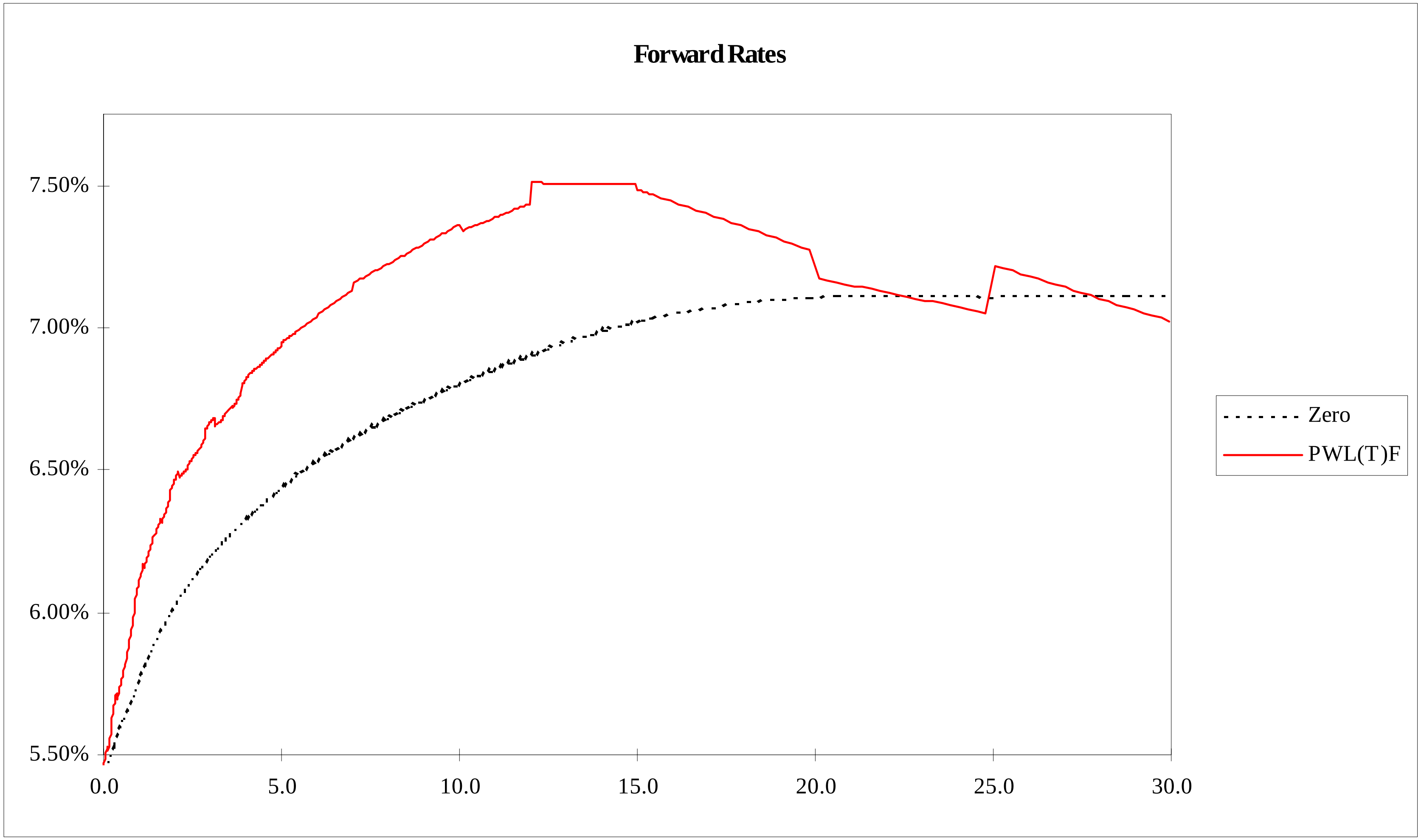
Functional Forms - PWLZ



Functional Forms - PWLF

- Choose break points (usually instrument maturities)
- Forwards linear between breaks
  - Generally more parameters than instruments
  - Twisted - set slope average of forwards on either side
  - Smoothed - minimize jumps and change in slopes
  - Two methods give virtually same results

Functional Forms - PWLF (twisted)



Risk and Hedging

- Risk measurement dependent on forward curve functional form
  - Constant forwards - risk interpolated approximately proportional to BPV
  - Linear zeros - risk interpolated approximately linearly
- Example - hedge 20 year with 10 & 30
  - Constant forwards - ratio of 22%/78%
  - Linear zero - ratio of 42%/58%

Addendum - Approximate Forwards

- Converts par yields (exact years) to PWCF
- Based on implicit function theorem
  - Par yield as function of forwards: $y_i = Y(f_1,...,f_i)$
  - Implies $dy_i = \Sigma_j\, a_{ij} df_j \qquad a_{ij} = \partial y_i / \partial f_j$
    $dY = A \cdot dF \qquad dF = A^{-1} \cdot dY$
  - Approx: $F \approx A^{-1} \cdot Y$

$$a_{ij} = \partial y_i / \partial f_j \approx [\partial PV_i / \partial f_j] / [dPV_i / dy_i]$$

based on $dPV_i = [dPV_i/dy_i]dy_i = \Sigma_j [\partial PV_i / \partial f_j] df_j$

$\Rightarrow \; dy_i = \Sigma_j \{[\partial PV_i / \partial f_j]/[dPV_i/dy_i]\} df_j$

approximate $[\partial PV_i / \partial f_j]$ by DV01 of forward bond

Addendum - cont'd

- Consider steeply down-ward sloping sterling swap curve, August 1999:

Matur	Par Y	Approx	True PWCF
5	6.74%	6.74%	6.74%
10	6.46%	6.08%	6.08%
20	5.98%	5.12%	5.14%
30	5.61%	3.91%	4.00%

- Works well considering steepness of curve
    - Error 9bp at 20-30 years

Conclusion

- Fitting the yield curve not difficult
- Big returns to a methodical approach
- Choose forward curve functional form based on how curve is used
- Choice of functional form matters