A DYNAMIC MODEL OF LABOR SUPPLY UNDER UNCERTAINTY*

T. Coleman
15 Grigg St.
Greenwich, CT 06830
tom@hilerun.org

J. Heckman
Department of Economics
University of Chicago
116 E 59th St.
Chicago, IL 60637
jheckman@uchicago.edu

Sept. 1981
Draft

* This research was supported by Department of Labor Contract No. 20-17-80-44 to the National Opinion Research Center. Organizations undertaking such projects under government sponsorship are encouraged to state their findings and express their judgments freely. Therefore, points of view or opinions stated in this document do not necessarily represent the official position of the Department of Labor.

I. Introduction

In this paper we discuss variations or additions to the standard wealth maximizing search model. The first is the distinction between the search and out-of-the labor force (OLF) states. An individual has the choice between (1) being employed at a given wage rate, (2) being out-of-the labor force (OLF) and receiving a wage offer next period. The usual search model consists of only states 1 and 2, i.e., an individual chooses between employment and search, with no opportunity for non-market activities.

The second addition is to make wage and on-wage offer distributions depend on both the state of the economy and current wage and non-wage offers. Lippman and McCall (1976a) have already formulated a model where wage offers depend on the state of the economy; our formulation is more general, but we do not claim any new results in this area. Correlated wage offers, however, do appear to be new, and provide interesting results. In particular, the reservation wage property no longer holds. Even given a particular state of the economy and a particular non-wage offer, there need not be a wage such that higher offers are accepted and lesser offers rejected. (Note that with correlated wages a moderate offer may be accepted - it gives no signal about next period's offer. A slightly higher wage, however, could be rejected because it signals a high probability of an even better offer next period.)

The third variation is to formulate these search models in continuous time and to allow wage offer arrivals in all states. We make the assumption that all events are independently Poisson distributed, so that the waiting time to an arrival is exponentially distributed. In other words wage offers arrive at random times, with the time being Poisson distributed. This formulation has intuitive appeal, since any discrete time formulation introduces arbitrary time units and it has the advantage of simplifying some of the econometric problems.

II. The Model

We will build the model in three stages. First we introduce the out of the labor force (OLF) state into the usual stationary search model. The distribution of new wage and non-wage offers will be constant over time, and there will be no change in the economy over time. This model exhibits most of the usual search model attributes; e.g., given a particular non-wage offer there is a unique reservation wage, and the value of search is increasing in the wage and non-wage offers. We give proofs of the existence and uniqueness of the value function, the reservation wage, and the other properties of this model.

In the second state we introduce non-stationarity. We take a relatively broad approach, allowing next period’s wage and non-wage offers to depend on this period’s offers, and on this and next period’s state of the economy. Many of the properties of the stationary search model do not carry through. We discuss properties of this model, but put the proofs of our properties in an appendix. The outline of the proofs are the same as for the stationary case, but technicalities make the non-stationarity case more difficult.

The third stage is to transform everything to continuous time, where offers arrive at random, independent, Poisson-distributed times. In addition we relax the constraint that wage offers only arrive in the search state. This is not a trivial change; in particular it opens the possibility of testing the existence of a search state separate from the employment and OLF states. The continuous time formulation is more tractable econometrically as well as being more realistic.

A. The Three State, Stationary Model

Individuals live forever and can be in one of three states:

1) Employed, receiving a known wage w per period until (voluntary) separation.
2) Out of the labor force (OLF), pursuing non-market activities with monetary remuneration of known n this period, and random N in the future.
3) Unemployed, paying a search cost of c this period to receive a random wage offer W new period.

Time is split into discrete units (of a month, say), and only one state, wage offer, or non-wage offer is allowed each period. All wage offers are known at the beginning of the period, any decision is made at the beginning of the period, any decision is made at the beginning, and money is earned or paid at the beginning of the period. A new non-wage offer N is received each period, but a new wage offer is received only after paying the search cost c. When employed, a worker receives a fixed, known wage w per period until he quits. A worker is allowed to switch from state 1 to 2 or 3, from 3 to 1 or 2, and from 2 to 3. (It will turn out that in the stationary case a worker will never quit to start searching, i.e., will never go from 1 to 3 directly, but he may go OLF for one period and then start searching, i.e., may go 1 to 2, and then 2 to 3.) The distribution of next period's wage offer, W, is $F(x)$, and the distribution of next period's non-wage offer, N, is $G(x)$; both $F(x)$ and $G(x)$ are known and unchanging.

We set up the problem by hypothesizing that there exists a continuous, bounded function of the current wage and non-wage offers, $V(w,n)$, that is the current value of the maximized expected lifetime earnings. This seems like an impossible problem, but using Bellman's principle we can make it simple. Denote the value of this period's maximized earnings as $V(w,n)$; then $\beta V(w,n)$ is the value (discounted at $\beta=1/(1+r)$) of receiving the same wage, w, next period and the new non-wage offer N. In other words $\beta V(w,N)$ is the value next period when employed this period. Similarly $\beta V(W,N)$ is next period's value when searching next period. Next period's value when OLF this period is $\beta V(0,N)$, because having no job is the same as a job with zero wage. Note that $\beta V(w,N)$, $\beta V(0,N)$, $\beta V(W,N)$ are all random variables; they depend on the random variables N, W. To get the expected value of each state we must add this period's return, and take the expectation over N and W, next period's random draws for non-wage and wage offers. Formally, the values of the three states are:

1) Being employed: $w + \beta EV(w,N) = w + \beta \int_0^\infty V(w,x)dG(x)$

2) Being OLF: $n + \beta EV(0,N) = n + \beta \int_0^\infty V(0,x)dG(x)$

3) Being unemployed (searching):

$$-c + \beta EV(W,N) = -c + \beta \int_0^\infty \int_0^\infty V(y,x)dF(y)dG(x)$$

In other words, the value of being in a particular state is this period's return plus the value of proceeding optimally tomorrow. The optimal value today will be the value of being in the best state. Mathematically, this says that V(w,n) must satisfy the following equation:

$$(1) \qquad V(w,n) = \max\{[w + \beta EV(w,N)],[n + \beta EV(0,N)],[-c + \beta EV(W,N)]\}$$

$$EV(w,N) = \int_0^\infty V(w,x)dG(x)$$

$$EV(0,N) = \int_0^\infty V(0,x)dG(x)$$

$$EV(W,N) = \int_{y=0}^\infty \int_{x=0}^\infty V(y,x)dF(y)dG(x)$$

It is shown in the appendix that (1) has a unique solution within the class of continuous functions.

Having constructed our model, which is summarized by equation (1) above, we can proceed to discuss some of its properties. The three-state model is much the same as the standard search model. In particular the value function, V(w,n) , is non-decreasing and convex in both its arguments. For a given value of n , i.e., a particular non-wage draw, there is a unique reservation wage offer, $w^*(n)$. Similarly for a given value of w , there is a unique reservation non-wage offer, $n^*(w)$. In addition both $w^*(n)$ and $n^*(w)$ are increasing in their arguments. All of this is a natural extension of the properties of standard two state models. For completeness, we will show how to prove these assertions.

First, we must reinterpret the right side of equation (1) as an operation T on the function v :

$$(2) \qquad (Tv)(w,n) = \max\{[w + \beta Ev(w,N)],[n + \beta Ev(0,N)],[-c + \beta Ev(W,N)]\} \; .$$

The function v(w,n) need not satisfy the equality in (1). In the appendix, however, we show that starting from any v^0 and applying T successively will lead to the V(w,n) which solves (1). In other words,

(3) $$V(w,n) = \lim_{t\to\infty}(T^t v^0)(w,n) \qquad \text{for any allowable } v^0(w,n)$$

where

$$(T^2 v^0)(w,n) = (Tv^1)(w,n)\,, \qquad v^1(w,n) \equiv (Tv^0)(w,n)$$

$$(T^3 v^0)(w,n) = (Tv^2)(w,n)\,, \qquad v^n(w,n) \equiv (Tv^1)(w,n)$$

…

$$(T^{n+1} v^0)(w,n) = (Tv^n)(w,n)\,, \qquad v^2(w,n) \equiv (Tv^{n-1})(w,n)$$

Properties

a) V(w,n) is increasing in (w,n) .

To see this, note that (3) tells us that starting from any allowable $v^0(w,n)$ leads to V(w,n) . If we start with a non-decreasing $v^0(w,n)$ and if the operation T preservers non-decreasingness, then the limit will also be non-decreasing. If $v^{n-1}(w,n)$ is non-decreasing in (w,n) , then

$$v^2(w,n) \equiv (Tv^{n-1})(w,n)$$

is obviously non-decreasing in (w,n) , from the definition of T in equation (2). Since $v^0(w,n) \equiv 0$ is a good starting function, and is obviously non-decreasing in (w,n) , V(w,n) is non-decreasing in (w,n) .

b) V(w,n) is convex in (w,n) .

A function f(x) is convex when

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha)f(x_2) \qquad \text{for all } 0 \le \alpha \le 1 \ .$$

The paragraph above implies that we need only show that a convex $v^n(w,n)$ gives a convex $v^{n+1}(w,n)$ and that an allowable $v^0(w,n)$ is convex. I.e. we must show

(4) $$v^n(\alpha w_1 + (1-\alpha)w_2, \alpha n_1 + (1-\alpha)n_2) \le \alpha v^n(w_1, n_1) + (1-\alpha)v^n(w_2, n_2)$$

implies

(5) $(Tv^n)(\alpha w_1 + (1-\alpha)w_2, \alpha n_1 + (1-\alpha)n_2) \le \alpha(Tv^n)(w_1, n_1) + (1-\alpha)(Tv^n)(w_2, n_2)$

for all $0 \le \alpha \le 1$. Write

$$\bar{w} = \alpha w_1 + (1-\alpha) w_2$$

$$\bar{n} = \alpha n_1 + (1-\alpha) n_2$$

and note that (4) also says (setting $w_1=w_2=w$, then $n_1=n_2=n$)

$$v^n(\alpha w_1 + (1-\alpha)w_2, n) \le \alpha v^n (w_1, n) + (1-\alpha)v^n(w_2, n)$$

$$v^n(w, \alpha n_1 + (1-\alpha)n_2) \le \alpha v^n (w, n_1) + (1-\alpha)v^n(w, n_2) \quad .$$

Then

$$(Tv^n)(\bar{w},\bar{n}) = \max\{[\bar{w} + \beta E v^n(\bar{w}, N)], [\bar{n} + \beta E v^n(0,N)], [-c + \beta E v^n(W,N)]\}$$

$$\le \max\left\{\begin{array}{l}[\alpha w_1 + (1-\alpha)w_2 + \beta E(\alpha v^n(w_1,N) + (1-\alpha)v^n(w_2,N))], \\ [\alpha n_1 + (1-\alpha)n_2 + \beta E(\alpha v^n(0,N) + (1-\alpha)v^n(0,N))], \\ [-\alpha c - (1-\alpha)c + \beta E(\alpha v^n(W,N) + (1-\alpha)v^n(W,N))]\end{array}\right\}$$

$$\le \alpha \max\{[w_1 + \beta E v^n(w_1,N)], [n_1 + \beta E v^n(0,N)], [-c + \beta E v^n(W,N)]\}$$
$$+ (1-\alpha)\max\{[w_2 + \beta E v^n(w_2,N)], [n_2 + \beta E v^n(0,N)], [-c + \beta E v^n(W,N)]\}$$

$$= \alpha(Tv^n)(w_1,n_1) + (1-\alpha)(Tv^n)(w_2,n_2) \quad .$$

The first inequality holds because $v^n(w,n)$ is convex in w . The second holds because the maximum of convex functions is itself convex.

c) For given values of n or w , there are unique reservation wage and non-wage offers, $w^*(n)$ and $n^*(w)$, respectively.

This is obvious from looking at equation (1) above. The reservation wage is defined to be that wage below which all offers are rejected, and above which all offers are accepted. Setting the value of a job equal to the value of searching, we get an equation defining the reservation wage for $n=0$, $w^*(0)$:

(6) $$w+\beta EV(w,N)=-c+\beta EV(W,N) \quad .$$

No job will be accepted for $W<w^*(0)$ since searching has more value. Similarly we can get the reservation non-wage offer for $w=0$, $n^*(0)$:

(7) $$n+\beta EV(0,N)=-c+\beta EV(W,N) \quad .$$

Since $EV(0,N)$ and $EV(W,N)$ are constants, and $EV(w,N)$ is increasing in w but is not a function of n, (6) and (7) define unique values. (There is one small caveat: the value of searching might be so low or the cost so high that any job is better than searching. In other words equations (6) and (7) may not have any solutions. If they do, however, they are unique.) For both w and n above $w^*(0)$ and $n^*(0)$, the choice is between taking a job and being out of the labor force (OLF). The reservation wage and non-wage offers will depend on the given value of n or w. $w^*(n)$ and $n^*(w)$ are both given by setting the value of a job equal to the value of OLF:

(8) $$w+\beta EV(w,N)=n+\beta EV(0,N)\ ,\ w>w^*(0)\ ,\ n>n^*(0) \quad .$$

Once again, if this has a solution, it is unique for $w^*(n)$ given n, or for $n^*(w)$ given w .

B. Three State, Non-Stationary Model

The model is essentially the same as in the stationary case, except that the random variables W and N are now functions of the current wage and non-wage offers. In addition, a new z variable is introduced to represent the state of the economy. This variable changes from period to period, and it also affects the random variables W and N. A higher value of z this period means the random wage and non-wage offers W and N are better, in that there is a higher probability of receiving a higher offer. A high z represents an active economy which has many high paying jobs, and also many lucrative opportunities out of the labor force. At this level of generality we don't specify exactly how z affects W and N, only that a higher z corresponds to more, better, offers. We also introduce the random variable Z, which is next period's state of the economy. Z is itself a function of z, this period's state of the economy, but it does not depend on the wage or non-wage offers. (We are taking a partial equilibrium approach to this model. All random variable, Z, W, and N, are exogenous to the decision process. In a full equilibrium model we would take into account the demand for labor and non-labor services, as well as individuals' supply decisions.)

To state things formally, Z, W, N are independent random functions over the real line $\Re$ and a probability space $(\Omega,\Im,P)$. In other words for real $\omega\in\Omega$

$$Z=Z(z,\omega)$$

$$W=W(w,z,\omega)$$

$$N=N(n,z,\omega) \quad .$$

The arguments z , w , n are restricted to lie in some finite interval $[0,M]$, and $Z(\cdot)$, $W(\cdot)$, $N(\cdot)$ map into the same interval $[0,M]$. This keeps the wage offers from ever becoming infinite or negative. To keep the interpretation of z as a measure of the strength of the economy we require that $W(w,z,\omega)$ and $N(n,z,\omega)$ are non-decreasing in z for each ω , w and n . This insures that the probability of receiving a wage (or non-wage) offer at least as high as some x is non-decreasing:

$$P[\omega : W(w,z,\omega) \geq x] = 1 - F_{w,z}(x)$$

is non-decreasing in z , for a given w . ($F_{w,z}(x)$ is the distribution function for W given particular values of w and z .) Since wage offers are non-negative this means the mean of $W(\cdot)$, for a given w , will increase with z :

$$\int_{\Omega} W(w.z_1, \omega) dP(\omega) \leq \int_{\Omega} W(w.z_2, \omega) dP(\omega)$$

for $z_1 \leq z_2$. What happens to higher moments about the mean depends on the particular function $W(w,z,\omega)$.

To maintain non-decreasingness of the value in the wage and non-wage offers we must have that $W(w,z,\omega)$ and $N(n,z,\omega)$ are non-decreasing in w and n , respectively. This has intuitive appeal if wages reflect productivity, and current wage offers serve as signals of productivity. This stretches our simple model somewhat, but does give some vague justification for assuming non-decreasingness of $W(w,z,\omega)$ and $N(n,z,\omega)$. Note, however, that the existence of an optimal policy and of the value function doesn't require this assumption. We make it because it insures increasingness of the value function in w and n .

The value function for this model will be a function of w , n , z : $V = V(w,n,z)$. The values of the three states are:

1) Being employed: $w + \beta EV(w, N, Z) = w + \beta \int_{\Omega} V\big(w, N(n, z, \omega), Z(z, \omega)\big) dP(\omega)$
2) Being OLF: $n + \beta EV(0, N, Z) = n + \beta \int_{\Omega} V\big(0, N(n, z, \omega), Z(z, \omega)\big) dP(\omega)$
3) Being unemployed (searching):
   $-c + \beta EV(W, N, Z) = -c + \beta \int_{\Omega} V\big(W(w, z, \omega), N(n, z, \omega), Z(z, \omega)\big) dP(\omega)$

The value of receiving a wage and non-wage offer of (w,n) when the state of the economy is z is the maximum of the three values:

(9) $V(w, n, z) = \max\{[w + \beta EV(w, N, Z)], [n + \beta EV(0, N, Z)], [-c + \beta EV(W, N, Z)]\}$.

where

$$EV(w,N,Z) = \int_{\Omega} V\big(w, N(n,z,\omega), Z(z,\omega)\big) dP(\omega)$$
$$= \int_{x=0}^{M} \int_{y=0}^{M} V(w,x,y) dG_{n,z}(x) dH_z(y)$$

$$EV(W,N,Z) = \int_{\Omega} V\big(W(w,z,\omega), N(n,z,\omega), Z(z,\omega)\big) dP(\omega)$$
$$= \int_{u=0}^{M} \int_{x=0}^{M} \int_{y=0}^{M} V(u,x,y) dF_{w,z}(u) dG_{n,z}(x) dH_z(y)$$

$F_{w,z}(u)$, $G_{n,z}(x)$, $H_z(y)$ are the distribution functions of $W(w,z,\omega)$, $N(n,z,\omega)$, and $Z(z,\omega)$ for given (w,n,z) .

In the same way that we could prove the existence and uniqueness of a solution to equation (1) for the stationary case, we can prove the existence and uniqueness of a solution to (9) within the class of continuous functions. The proof of this, together with proof of convexity and increasingness of the solution to (9), is found in the appendix.

Taking the existence and uniqueness of $V(w,n,z)$ as given we can easily see that the reservation wage property of simple search models does not hold. Look, for example, at the condition for indifference between taking a job and continuing search (for a given n):

(10) $w + \beta EV\big(w, N(n,z,\omega), Z(z,\omega)\big) = -c + \beta EV\big(W(w,z,\omega), N(n,z,\omega), Z(z,\omega)\big)$.

Both sides are increasing functions of w , even after we take the expectation. Thus we don't have any guarantee that (10) has a unique solution. It could happen that there are three solutions, so that all wages below w_1 are rejected, those between w_1 and w_2 accepted, those between w_2 and w_3 rejected, and those above w_3 accepted. This is a direct result of correlated wage offers. This situation might result if low wages (below w_2) all had significantly higher probability of getting even better offers. Then any offer below w_2 would lead to normal search model behavior, with some reservation wage w_1 . Any offer above w_2 might lead a worker to stay out of the market in expectation of a yet better offer. Thus we get a set of wages, between w_2 and w_3 , which are rejected.

The same kind of argument holds for non-wage offers. There is no reservation non-wage offer as long as we allow correlated non-wage offers.

We can also look at the employment, OLF, search choice as a function of z , for given w and n . There is no one value of z at which a worker is indifferent between work and search, search and OLF, and work and OLF. In the same way that correlation in wage offers may make it worthwhile to sit out in the expectation of higher offers next period, certain states of the economy may signal a high probability of even better times next period. This may lead to acceptance of low wage offers for low z, rejection of low offers for intermediate z, and then acceptance of the same low offers for yet higher z. In the intermediate range workers may sit out in the expectation of a better state of the economy and thus better offer distributions next period.

By dropping the correlation in wage and non-wage offers we get something that looks like the Lippman and McCall (1976a) model. The functional equation for an infinite lived individual is

$$(11)\ V(w,n,z) = \max\{[w + \beta EV(w,N,Z)],[n + \beta EV(0,N,Z)],[-c + \beta EV(W,N,Z)]\}$$

$$W = W(z,\omega)$$

$$N = N(z,\omega)$$

$$Z = Z(z,\omega)$$

The equation corresponding to (1), which defines the points of indifference between working and searching, is now

$$(12)\qquad w + \beta EV(w, N(z,\omega), Z(z,\omega)) = -c + \beta EV(W(z,\omega), N(z,\omega), Z(z,\omega))$$

which defines a unique w. Call this $w^*(0,z)$, and define $n^*(0,z)$ by a corresponding equation. Then for both $w>w^*(0,z)$ and $n>n^*(0,z)$ and for given z, we can define $w^*(n,z)$ and $n^*(w,z)$ implicitly by

$$(13)\qquad w + \beta EV(w, N(z,\omega), Z(z,\omega)) = n + \beta EV(0, N(z,\omega), Z(z,\omega))\ .$$

These are all uniquely determined, but $w^*(n,z)$ and $n^*(w,z)$ are not increasing in z. Because of correlation in states of the economy and dependence on the offer distributions on z, both sides of (12) and (13) are increasing in z. A low state of the economy might have lower reservation wage than an intermediate state, but the intermediate state might have a higher reservation wage than a high state.

C. Three State Model – Continuous Time

To begin the continuous time model we will return to our stationary assumption: W and N are random variables which don't change over time and don't depend on any other variables. Time is continuous, and wage offers can arrive at any time. The offers themselves are discrete and arrive at random times. We will assume that the offer arrival times are Poisson distributed, so that the time until a next arrival does not depend on the wait since last arrival.

One major change from the discrete time model (although this could have been incorporated in the discrete time model) is that wage offers will be allowed to arrive in all states. In other words an individual in the job state will receive offers without having to quit. What will distinguish the job, OLF, and search states is the rate of arrival of offers. The search state will presumably have the highest rate of arrival of new job offers, with job and OLF lower. The arrival rate for non-wage offers we will denote by μ , and for wage offers in the employment, out of the labor force, and search states by λ_1 , λ_2 , and λ_3 .

To derive the functional equation for the value, we will first derive functional equations for each of the states separately. The values for job, OLF, and search we will denote by $V_1(w)$, $V_2(w)$, and V_3 . To calculate the value, we need to know two things. First, when does a new offer arrive, and second, what kind of offer is it. For the job state, two kinds of offers, non-wage and wage, arrive. The non-wage offers arrive at rate μ , the wage offers are rate λ_1 . The time of arrival of some offer is the minimum of the time of arrival of either non-wage or wage offers, and so arrives at rate $\mu+\lambda_1$. In other words, since the density of arrival times for non-wage offers is

$$\mu e^{-\mu t}$$

and for wage offers is

$$\lambda_1 e^{-\lambda 1 t} \quad ,$$

the density of some arrival is

$$(\mu+\lambda_1)e^{-(\mu+\lambda 1)t} \quad .$$

We also know the density of non-wage offers conditional on the non-wage offering first. Calling t_μ the time of arrival of a non-wage offer, and t_1 the time of arrival of a wage offer, the conditional distribution is

$$F_{t_\mu}(t) = P\left[t_\mu \le t \text{ and } t_1 > t\right] = \left[1 - e^{-\mu t}\right] e^{-\lambda_1 t} \quad .$$

The conditional density is

$$f_{t_\mu}(t) = \mu e^{-(\mu+\lambda_1)t} \quad . \tag{14}$$

Similarly, the density for t , conditional that the wage offer arrives first, is

$$\lambda_1 e^{-(\mu+\lambda_1)t} \quad . \tag{15}$$

In the job state, an individual continues in the state until a new offer arrives. At that time he can choose to switch to the OLF state (if a non-wage offer arrives) or switch to a better job (if a wage offer arrives). We implicitly assume that workers are not allowed to switch to a better state except at arrivals of new offers. For example, a worker who happens to be at a low paying job might desire to switch to search, but we will not allow him to do so except when new offers arrive. This will not affect the value of the job state when the job state is optimal, and that is all we are interested in. One minor problem arises when comparing the job and OLF states at zero wage and non-wage levels. It is possible, for certain sets of parameters, for a zero-wage job to be of higher value than a zero, or even slightly positive, non-wage offer. We will allow individuals to switch into the zero-wage job, although we will generally pick parameters so the situation doesn't arise ($\lambda_1 \le \lambda_2$).

a) Being paid wage w until time τ, then receiving a new wage offer. The worker can either continue at w, take the new W (if it is higher) or switch to search. This is all conditional on the wage offer arriving first.

$$\int_0^\tau e^{-ru} w du + e^{-r\tau} \max[V_1(\max(W,w)), V_3] = \frac{w}{r}\left[1-e^{-r\tau}\right] + e^{-r\tau} \max[V_1(\max(W,w)), V_3]$$

b) Being paid wage w until time τ, then receiving a new non-wage offer. The worker can then either continue working (and receive $V_1(w)$) or switch to the OLF state (and receive $V_2(N)$). This is conditional on the non-wage offer arriving first.

$$\int_0^\tau e^{-ru} w du + e^{-r\tau} \max[V_1(w), V_2(N), V_3] = \frac{w}{r}\left[1-e^{-r\tau}\right] + e^{-r\tau} \max[V_1(w), V_2(N), V_3]$$

Using the conditional densities for wage and non-wage offers arriving first, and taking expectations separately over τ, W, and N (all are independent) we get

$$\begin{aligned} V_1 &= \frac{w}{r}\int_0^\infty \left[1-e^{-r\tau}\right](\mu+\lambda_1)e^{-(\mu+\lambda_1)\tau} d\tau \\ &+ \left[\int_0^\infty e^{-r\tau}\lambda_1 e^{-(\mu+\lambda_1)\tau} d\tau\right] E\max[V_1(\max(W,w)), V_3] \\ &+ \left[\int_0^\infty e^{-r\tau}\mu e^{-(\mu+\lambda_1)\tau} d\tau\right] E\max[V_1(w), V_2(N), V_3] \end{aligned}$$

(16) $$V_1(w) = \frac{1}{\mu+\lambda_1+r}\left\{w + \lambda_1 E\max[V_1(\max(W,w)), V_3] + \mu E\max[V_1(w), V_2(N), V_3]\right\}.$$

For the OLF state, the choices facing an individual are

a) Receiving non-wage benefits n until τ, then receiving a new non-wage offer, N. The individual can then choose between search, V_3, and continuing in the OLF state, $V_2(N)$. This is conditional on the non-wage offer arriving first.

$$\int_0^{\tau} e^{-ru} n du + e^{-r\tau} \max[V_2(N), V_3, V_1(0)] = \frac{n}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} \max[V_2(N), V_3, V_1(0)]$$

b) Receiving non-wage benefits n until τ , then receiving a new wage offer. The choice is between continuing in OLF $V_2(N)$, switching to the job, $V_1(W)$, or switching to search, V_3 . This is conditional on the wage offer arriving first.

$$\int_0^{\tau} e^{-ru} n du + e^{-r\tau} \max[V_1(W), V_2(n), V_3] = \frac{w}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} \max[V_1(W), V_2(n), V_3]$$

Using the conditional densities we get

$$V_2 = \frac{n}{r}\int_0^{\infty}\left[1 - e^{-r\tau}\right](\mu + \lambda_2) e^{-(\mu+\lambda_2)\tau} d\tau$$
$$+ \left[\int_0^{\infty} e^{-r\tau} \mu e^{-(\mu+\lambda_2)\tau} d\tau\right] E \max[V_2(N), V_3, V_1(0)]$$
$$+ \left[\int_0^{\infty} e^{-r\tau} \lambda_2 e^{-(\mu+\lambda_2)\tau} d\tau\right] E \max[V_1(W), V_2(n), V_3]$$

(17) $V_2(n) = \frac{1}{\mu + \lambda_2 + r}\{n + \mu E \max[V_2(N), V_3, V_1(0)] + \lambda_2 E \max[V_1(W), V_2(n), V_3]\}$.

For the search state, the choices facing an individual are

a) Paying the cost c per unit time until τ , then receiving a new non-wage offer, N . The individual can then choose between search, V_3 , and switching to OLF, $V_2(N)$. This is conditional on the non-wage offer arriving first.

$$-\int_0^{\tau} c e^{-ru} du + e^{-r\tau} \max[V_2(N), V_3, V_1(0)] = -\frac{c}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} \max[V_2(N), V_3, V_1(0)]$$

b) Paying c per unit time until τ , then receiving a new wage offer W . The individual can then choose between search, V_3 , and the job, $V_1(W)$. This is conditional on the wage offer arriving first.

$$-c\int_0^{\tau} e^{-ru} du + e^{-r\tau} \max[V_1(W), V_3] = -\frac{c}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} \max[V_1(W), V_3]$$

Using the conditional densities for arrival of the non-wage and wage offers first, we get

$$V_3 = -\frac{c}{r}\int_0^\infty \left[1-e^{-r\tau}\right](\mu+\lambda_3)e^{-(\mu+\lambda_3)\tau}d\tau$$
$$+\left[\int_0^\infty e^{-r\tau}\mu e^{-(\mu+\lambda_3)\tau}d\tau\right]E\max\left[V_2(N),V_3,V_1(0)\right]$$
$$+\left[\int_0^\infty e^{-r\tau}\lambda_3 e^{-(\mu+\lambda_3)\tau}d\tau\right]E\max\left[V_1(W),V_3\right]$$

(18) $$V_3 = \frac{1}{\mu+\lambda_3+r}\left\{-c+\mu E\max\left[V_2(N),V_3,V_1(0)\right]+\lambda_3 E\max\left[V_1(W),V_3\right]\right\} .$$

Putting all three values together, we can say that the value to the individual of this search problem is

$$V(w,n) = \max\left[V_1(w),V_2(n),V_3\right]$$

(19) $$= \max\left\{\begin{array}{l}\frac{1}{\mu+\lambda_1+r}\left[w+\lambda_1 E\max\left[V_1(\max(W,w)),V_3\right]+\mu E\max\left[V_1(w),V_2(N),V_3\right]\right], \\ \frac{1}{\mu+\lambda_2+r}\left[n+\mu E\max\left[V_2(N),V_3,V_1(0)\right]+\lambda_2 E\max\left[V_1(W),V_2(n),V_3\right]\right], \\ \frac{1}{\mu+\lambda_3+r}\left[-c+\mu E\max\left[V_2(N),V_3,V_1(0)\right]+\lambda_3 E\max\left[V_1(W),V_3\right]\right]\end{array}\right\}$$

Note, however, that when the job is optimal,

(20) $$V(w,n)=V(w,0)=V_1(w)=\frac{1}{\mu+\lambda_1+r}\left[w+\lambda_1 EV(\max(W,w),0))+\mu EV(w,N)\right] .$$

Similarly, when OLF is optimal,

(21) $$V(w,n)=V(0,n)=V_2(n)=\frac{1}{\mu+\lambda_2+r}\left[n+\mu EV(0,N)+\lambda_2 EV(W,n)\right]$$

and when search is optimal

(22) $$V(w,n)=V(0,0)=V_2=\frac{1}{\mu+\lambda_3+r}\left[-c+\mu EV(0,N)+\lambda_3 EV(W,0)\right] .$$

Thus V(w,n) can be expressed as

$$\text{(23)} \qquad V(w,n) = \max\left\{\begin{array}{l} \frac{1}{\mu+\lambda_1+r}\left[w+\mu EV(w,N)+\lambda_1 EV(\max(W,w),0)\right], \\ \frac{1}{\mu+\lambda_2+r}\left[n+\mu EV(0,N)+\lambda_2 EV(W,n)\right], \\ \frac{1}{\mu+\lambda_3+r}\left[-c+\mu EV(0,N)+\lambda_3 EV(W,0)\right] \end{array}\right\}$$

Equation (23) can be put in a slightly different form by rearranging equations (20)-22). Start with equation (20) and add $(\lambda_2+\lambda_3)V(w,0)/(\mu+\lambda_1+r)$ to both sides:

$$V(w,0)+\frac{\lambda_2+\lambda_3}{\mu+\lambda_1+r}V(w,0)$$
$$=\frac{1}{\mu+\lambda_1+r}\left[w+\lambda_1 EV(\max(W,w),0)+\mu EV(w,N)+(\lambda_2+\lambda_3)V(w,0)\right]$$

$$V(w,0)=\frac{1}{\mu+\lambda_1+\lambda_2+\lambda_3+r}\left[w+\mu EV(w,N)+\lambda_1 EV(\max(W,w),0)+(\lambda_2+\lambda_3)V(w,0)\right]$$

Performing similar algebra on (21) and (22), gives

$$\text{(24)} \qquad V(w,n)=\frac{1}{\eta+r}\max\left\{\begin{array}{l} \left[w+\mu EV(w,N)+\lambda_1 EV(\max(W,w),0))+(\lambda_2+\lambda_3)V(w,0)\right], \\ \left[n+\mu EV(0,N)+\lambda_2 EV(W,n)+(\lambda_1+\lambda_3)V(0,n)\right], \\ \left[-c+\mu EV(0,N)+\lambda_3 EV(W,0)+(\lambda_1+\lambda_2)V(0,0)\right] \end{array}\right\}$$

where $\eta \equiv \mu+\lambda_1+\lambda_2+\lambda_3$.

Equation (24) can be derived in one step by an alternative method. In the job state, the following four events, with their associated rate of arrivals, could occur:

a) A new non-job (OLF) offer, N , arrives, with rate μ .
b) A new job offer, W , arrives, with rate λ_1 .
c) A new job offer for the OLF state arrives, with rate λ_2 . Since we are in the state; i.e., the same wage w .
d) A new job offer for the search state arrives, with rate λ_3 . Since we are in the job state, this is no change in the state; i.e., the same wage w .

Calling the time of first arrival τ , the conditional densities for the events a), b), c), d) to occur first are (see equation (14))

$$\mu e^{-\eta\tau}$$

$$\lambda_1 e^{-\eta\tau}$$

$$\lambda_2 e^{-\eta\tau}$$

$$\lambda_3 e^{-\eta\tau}$$

where $\eta \equiv \mu + \lambda_1 + \lambda_2 + \lambda_3$. The expected value for each event is, conditional on it occurring first

a) $\frac{w}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} EV(w, N)$

b) $\frac{w}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} EV\left(\max(W, w), 0\right)$

c) $\frac{w}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} V(w, 0)$

d) $\frac{w}{r}\left[1 - e^{-r\tau}\right] + e^{-r\tau} V(w, 0)$

(We are implicitly assuming that once a worker takes a job he cannot go back to the OLF state. This won't matter if the job is the optimal state since a worker would not go to OLF, and if the job is not optimal we don't really care what the value of the job state is.) Using the conditional densities above to take the expectation over τ, we get for the job state

$$\frac{1}{\eta + r}\left[w + \mu EV(w, N) + \lambda_1 EV\left(\max(W, w), 0\right) + (\lambda_2 + \lambda_3) V(w, 0)\right] \ .$$

Similarly for the OLF and search states we get

$$\frac{1}{\eta + r}\left[n + \mu EV(0, N) + \lambda_2 EV(W, n) + (\lambda_1 + \lambda_3) V(0, n)\right]$$

$$\frac{1}{\eta + r}\left[-c + \mu EV(0, N) + \lambda_3 EV(W, 0) + (\lambda_1 + \lambda_2) V(0, 0)\right]$$

Thus,

$$V(w, n) = \frac{1}{\eta + r} \max \begin{Bmatrix} \left[w + \mu EV(w, N) + \lambda_1 EV\left(\max(W, w), 0\right) + (\lambda_2 + \lambda_3) V(w, 0)\right], \\ \left[n + \mu EV(0, N) + \lambda_2 EV(W, n) + (\lambda_1 + \lambda_3) V(0, n)\right], \\ \left[-c + \mu EV(0, N) + \lambda_3 EV(W, 0) + (\lambda_1 + \lambda_2) V(0, 0)\right] \end{Bmatrix}$$

The existence and uniqueness of V(w,n) is established in the appendix, as is the increasingness in (w,n) . It should be clear that the model summarized by equation (24) has a reservation wage for given n , and a reservation non-wage offer for given w .

We can introduce non-voluntary lay-offs in the job state by introducing layoffs that occur at random, Poisson distributed times. Following an argument analogous to that above, we get the expected value of the three states as:

1) Job:
$$\frac{1}{\eta+\delta+r}\left[w+\mu EV(w,N)+\lambda_1 EV\left(\max(W,w),0\right)+\left(\lambda_2+\lambda_3\right)V(w,0)+\delta V(0,n)\right] \quad .$$
2) OLF: $\frac{1}{\eta+\delta+r}\left[n+\mu EV(0,N)+\lambda_2 EV(W,n)+\left(\lambda_1+\lambda_3+\delta\right)V(0,n)\right]$
3) Search: $\frac{1}{\eta+\delta+r}\left[-c+\mu EV(0,N)+\lambda_3 EV(W,n)+\left(\lambda_1+\lambda_2+\delta\right)V(0,n)\right]$

and the value function as the maximum of the three. Note that because of the layoffs, where a worker is required to fall back on the non-wage offer, n enters into the job state in a non-trivial way. This also means that there might be no unique reservation non-wage offer. For a given n , however, there will still be a unique reservation wage offer.

Just as the discrete time model can be extended by adding correlated wage offers and a state of the economy variable, the continuous time model can be extended. It is largely, however, a matter of notation, since we just make W and N random functions which depend on the current w and n . Formally, let a probability space $(\Omega,\Im,P)$ be given and

$$W = W(w,z,\omega) : [0,M] \times [0,M] \times \Omega \rightarrow [0,M]$$

$$N = N(n,z,\omega) : [0,M] \times [0,M] \times \Omega \rightarrow [0,M]$$

$$Z = Z(z,\omega) : [0,M] \times \Omega \rightarrow [0,M] \quad .$$

We will assume that $W(\cdot)$, $N(\cdot)$, $Z(\cdot)$ are all increasing in w , n , z for each $\omega\in\Omega$. New wage and non-wage offers arrivals are Poisson as before (wage offers with means λ_1 , λ_2 , λ_3 in the employment, OLF, and search state, non-wage offers with mean μ). New states of the economy arrive at rate ν . The functional fixed point equation is

(25)

$$V(w,n,z)=\frac{1}{\eta+r}\max\left\{\begin{array}{l}\left[\begin{array}{l}w+\mu EV(w,N,z)+\lambda_1 EV\left(\max(W,w),n,z\right)\\+\left(\lambda_2+\lambda_3\right)V(w,n,z)+\delta V(0,n,z)+\nu EV(w,n,Z)\end{array}\right],\\\left[\begin{array}{l}n+\mu EV(0,N,z)+\lambda_2 EV(W,n,z)\\+\left(\lambda_1+\lambda_3+\delta\right)V(0,n,z)+\nu EV(0,n,Z)\end{array}\right],\\\left[\begin{array}{l}-c+\mu EV(0,N,z)+\lambda_3 EV(W,n,z)\\+\left(\lambda_1+\lambda_2+\delta\right)V(0,n,z)+\nu EV(0,n,Z)\end{array}\right]\end{array}\right\}$$

$\eta\equiv\mu+\lambda_1+\lambda_2+\lambda_3+\delta+\nu$.

APPENDIX A

We prove assertions in this appendix to keep the text uncluttered and to allow more rigorous proofs. In some cases we have repeated proofs from the text for completeness. The methods used here are simple and well known to mathematicians, but because they have not been widely used in the search literature we have laid out some of the proofs in more detail than is absolutely necessary. The concept we use, fixed point functional equations, is not too complex and the mechanics of proving existence, uniqueness, and other properties is simple.

II. A. Three State Stationary Model

<u>Preliminaries</u>

The basic equation for this model is (1), reproduced here as equation (A.1).

(A.1) $V(w,n) = \max\{[w+\beta EV(w,N)],[n+\beta EV(0,N)],[-c+\beta EV(W,N)]\}$

$$EV(w,N) = \int_0^\infty V(w,x)dG(x)$$

$$EV(0,N) = \int_0^\infty V(0,x)dG(x)$$

$$EV(W,N) = \int_{y=0}^\infty \int_{x=0}^\infty V(y,x)dF(y)dG(x)$$

Equation (A.1) is the basic equation of our model. The problem is to prove that there is actually function V(w,n) for which (A.1) holds; so far we have only assumed such a function exists. To tackle this problem we will look at (A.1) in a slightly different light. For any continuous, bounded function $v(\cdot,\cdot)$; define the operation T by

(A.2) $(Tv)(w,n) \equiv \max\{[w+\beta Ev(w,N)],[n+\beta Ev(0,N)],[-c+\beta Ev(W,N)]\}$.

$$Ev(w,N) = \int_0^\infty v(w,x)dG(x)$$

$$Ev(0,N) = \int_0^\infty v(0,x)dG(x)$$

$$Ev(W,N) = \int_{y=0}^{\infty} \int_{x=0}^{\infty} v(y,x) dF(y) dG(x)$$

First, we show that for any $v^0(\cdot,\cdot)$, $(Tv)(\cdot,\cdot)$ is a new function, $v^1(\cdot,\cdot)$, which is also continuous and bounded, given that the range of W, N are bounded, i.e., that we don't allow infinite wage and non-wage offers. This leads to

> Assumption 1 – The values of W, N, the random wage and non-wage draws, are bounded between zero and $M<\infty$. I.e. $0 \le W \le M$, $0 \le N \le M$, $M<\infty$.

T is an operator from the space of continuous functions back to the same space. Thus equation (A.1) can also be written as a fixed point functional equation

(A.3) $$v(w,n) = (Tv)(w,n)$$

In essence this is no more complicated than an implicit function (say $x = e^x + 1$), which may have zero, one, or many solutions. The real difference, and this turns out to be relatively unimportant, is that $e^x + 1$ takes real numbers, x, and maps them back into real numbers, while the operator T takes functions and maps them back into functions. By defining the appropriate measure of distance between the two functions $v^0(.,.)$ and $v^1(.,.)$, we can forget that v^0 and v^1 are functions and treat them almost as if they were points in space. For continuous functions the correct measure of distance is called the sup-norm, the supremum of the absolute of the difference between the function values:

(A.4) $$d\left(v^0, v^1\right) = \left\| v^0 - v^1 \right\| = \sup_{\substack{0 \le x \le M \\ 0 \le y \le M}} \left| v^0(x,y) - v^1(x,y) \right| \quad .$$

Under this norm, the space of continuous, bounded function over the finite range $[0,M]$ is closed (all Cauchy sequences converge), and we can use Brauer's fixed point theorem.

<u>Brauer's fixed point theorem:</u>

(A.5) $$d\left(Tv^0, Tv^1\right) \le \beta d\left(v^0, v^1\right) \qquad \text{for any } v^0,\ v^1,\ 0<\beta<1$$

implies the functional fixed point equation $v=Tv$ has a unique solution, and the solution is

(A.6) $$v^* = \lim_{t \to \infty} T^t v^0 \qquad \text{for any } v^0$$

where $T^2v=T(Tv)$, $T^3v=T(T^2v)$, etc. (See, e.g., Wouk (1979).)

The method just outlined has a very simple intuitive and economic interpretation. We can use the operator T defined in (A.2) to generate a sequence of value functions $v^0, v^1, v^2, \ldots, v^N$, where $v^{i+1} = Tv^i$ and $v^0(w,n) = 0$. v^0 has the economic interpretation of the value of not being allowed any choice, in short of being dead. v^1 is then the value of having one period left to live, v^2 the value of two more periods, and so on up to v^N being the value of living for N periods. Equation (A.5) gives a condition under which the sequence $v^0, v^1, \ldots$ will converge to some unique v^* which satisfies (A.3). (It is actually stronger, since it says that a sequence starting from any initial v^0, not just $v^0(w,n) \equiv 0$, will converge.) This procedure is called backwards induction, and is very useful for proving specific properties of the value function.

For example, we can assert: Equation (A.3) (or equation (A.1)) has a unique solution within the class of continuous functions $v : [0,M] \times [0,M] \rightarrow \Re$.

We use Blackwell's conditions (from Blackwell (1965)). Although Blackwell limits himself to continuous bounded functions, his method does not require this. For completeness we will reproduce his proof here. His theorem 5 (paraphrased) is

> If an operator, T, from the space of continuous functions over a finite region of $\Re^2$, $[0,M] \times [0,M]$, to the real line $\Re$, satisfies
>
> a) monoticity: $v^0 \le v^1 \Rightarrow T(v^0) \le T(v^0)$
> b) $T(v^0 + k) = T(v^0) + \beta k$, $0 \le \beta \le 1$, k constant,
>
> then T is a contraction mapping with
>
> $$d(Tv^0, Tv^1) = \|Tv^0 - Tv^1\| \le \beta \| v^0 - v^1 \| \quad .$$
>
> PROOF:
>
> $v^0(w,n) \le v^1(w,n) + \| v^0 - v^1 \|$ by the definition of the sup-norm.
>
> $(Tv^0)(w,n) \le (Tv^1)(w,n) + \beta \| v^0 - v^1 \|$ by conditions a) and b)
>
> $v^1(w,n) \le v^0(w,n) + \| v^0 - v^1 \|$ by definition
>
> $(Tv^1)(w,n) \le (Tv^0)(w,n) + \beta \| v^0 - v^1 \|$ by a) and b)
>
> Using these two inequalities implies
>
> $|(Tv^0)(w,n) - (Tv^1)(w,n)| \le \beta \| v^0 - v^1 \|$ and so
>
> $\|Tv^0 - Tv^1\| \le \beta \|v^0 - v^1\|$.

To allow for greater generality later, we will allow more general random variables than just W, N. In other words, we will prove the existence and uniqueness of a function which satisfies

$$V(w,n) = \max\{[w + \beta EV(X_1)], [n + \beta EV(X_2)], [-c + \beta EV(X_3)]\} \quad ,$$

where the vector valued random variables X_1 , X_2 , X_3 are restricted to the rectangle $[0,M] \times [0.M]$. In our case

$$X_1 = (w,N)$$

$$X_2 = (0,N)$$

$$X_3 = (W,N) \quad .$$

Another possibility might be

$$X_2 = \begin{cases} (W,N) \text{ with probability } p \\ (0,N) \text{ with probability } (1-p) \end{cases}$$

with X_1 and X_3 remaining the same. This would correspond to receiving wage offers in the OLF state, but only with a probability p .

To use Blackwell's conditions, we just have to show a) and b) above. Monotonicity is simple, since $v^0(w,n) \le v^1(w,n)$ implies

$$\begin{aligned}(Tv^0)(w,n) &= \max\{[w + \beta Ev^0(X_1)], [n + \beta Ev^0(X_2)], [-c + \beta Ev^0(X_3)]\} \\ &\le \max\{[w + \beta Ev^1(X_1)], [n + \beta Ev^1(X_2)], [-c + \beta Ev^1(X_3)]\} \\ &= (Tv^1)(w,n)\end{aligned}$$

Showing $T(v+k)(w,n) = (Tv)(w,n) + \beta k$, k constant, is simple, since

$$\begin{aligned}(Tv + k)(w,n) &= \max\{[w + \beta Ev(X_1) + \beta k], [n + \beta Ev(X_2) + \beta k], [-c + \beta Ev(X_3) + \beta k]\} \\ &= \max\{[w + \beta Ev(X_1)], [n + \beta Ev(X_2)], [-c + \beta Ev(X_3)]\} + \beta k \\ &= (Tv)(w,n) + \beta k\end{aligned}$$

Thus T is a contraction mapping. QED

Unbounded Support

We have assumed that both W and N are restricted to the interval [0,M] , so that each has bounded support. This has obvious disadvantages when turning to empirical applications; distributions commonly used in empirical applications have unbounded support. The reason we have not extended our analysis to unbounded W and N is a seemingly technical annoyance. The space of continuous functions over a finite interval is closed under the sup-norm used above. Extending the analysis to V(w,n) where (w,n) range over an unbounded interval does not seem to be allowed. Our first attempt at circumventing this problem was to look at the space of integrable functions, and using the L^1 or L^2 norm. This does not appear to work, however, since we have found an example where

$$\sup_{w\in[0,M]}\left|Tv^1(w)-Tv^2(w)\right|\leq\beta\sup_{w\in[0,M]}\left|v^1(w)-v^2(w)\right|$$

but

$$\int\left|Tv^1(w)-Tv^2(w)\right|dF(w)>\beta\int\left|v^1(w)-v^2(w)\right|dF(w)$$

where

$$\left(Tv^1\right)(w,n)\equiv\max\left\{\left[w+\beta v^1(w)\right],\left[-c+\beta Ev^1(W)\right]\right\}\ .$$

In other words we have found functions such that the distances between Tv^1 and Tv^2 is less than the distance between v^1 and v^2 when measured in the sup-norm but is not when measured in the L^1 norm. (The example is to use the interval [0,2], with W uniformly distributed. Pick

$$v^1(w)=0$$

$$v^2(w)=\begin{cases}0 & w\in[0,1]\\ \dfrac{w-1}{\beta} & w\in(1,2]\end{cases}$$

$$c=1/8$$

Then

$$\int_0^2\left|Tv^1(w)-Tv^2(w)\right|dF(w)=0.2539$$

$$\beta\int_0^2\left|v^1(w)-v^2(w)\right|dF(w)=0.25\ \ .\ )$$

We have found a trick that, while not very elegant, seems to work. Instead of treating the value function as a function of (w,n) and the random variables (W,N) (with distributions F(x) , G(y)), treat it as a function of an underlying probability space. Use the space $(\Omega,\Im,P)$ = ((0,1), Borel sets of (0,1), Lebesgue measure). Make W,N random variables of $\omega\in\Omega$. Since any distribution F(x) on $\Re^1$ can be represented by a random variable on $(\Omega,\Im,P)$, we can represent any random variable W with distribution F(x) by a random function $W(\omega)$. Thus we can think of $w(\cdot)$ and $n(\cdot)$ as functions of t_1 and t_2 , $w(t_1)$, $n(t_1)$. We now rewrite (A.1) as

$$V(t_1,t_2)=\max\left\{\begin{matrix}\left[w(t_1)+\beta\int V(t_1,\omega_2)d\omega_2\right],\left[n(t_2)+\beta\int V(\bar{t}_1,\omega_2)d\omega_2\right],\\ \left[-c+\beta\int\int V(\omega_1,\omega_2)d\omega_1 d\omega_2\right]\end{matrix}\right\}$$

where $\bar{t}_1 = \text{any}\, t$, for which $w(t_1) = 0$. Now $t_1 \in (0,1)$, $t_2 \in (0,1)$, so $V(t_1,t_2)$ is a function over $(0,1)\times(0,1)$. (There is one small problem – $w(t_1)$ and $n(t_2)$ must be continuous function of t_1 and t_2 so that $V(t_1,t_2)$ will be continuous.) The analysis of (A.1) or (A.3) still holds. The proof holds with only change of notation (writing $w(t_1)$ instead of w).

The condition that $V(t_1,t_2)$ be defined $(<\infty)$, at least for values $(t_1,t_2) : w(t_1)<\infty$, $n(t_2)<\infty$, is that $w(t_1)$ and $n(t_2)$ are both integrable. Given a $v^0(t_1,t_2)$ that is integrable,

(A.4) $$\int_0^1\int_0^1 v^0(t_1,t_2)dt_1dt_2<\infty$$

and also

$$\int_0^1 v^0(t_1,t_2)dt_2<\infty \quad \forall t_1 \text{ s.t. } w(t_1)<\infty.$$

(A.5) $$\int_0^1 w(t_1)dt_1<\infty \quad \int_0^1 n(t_2)dt_2<\infty$$

then $(Tv^0)(t_1,t_2)$ is integrable, and $(Tv^1)(t_1,t_2)<\infty$ $\forall(t_1,t_2)$ such that $w(t_1)<\infty$, $n(t_2)<\infty$.

<u>Proof</u>:
Given the condition (A.4), $\int v^0(t_1,\omega_2)d\omega_2<\infty$ $\forall t_1$ s.t. $w(t_1)<\infty$ $\int v^0(\omega_1,\omega_2)d\omega_1 d\omega_2<\infty$. So when $w(t_1)<\infty$, $n(t_1)<\infty$, $(Tv^0)(t_1,t_2)<\infty$. In addition

$$\begin{aligned}(Tv^0)(t_1,t_2)dt_1dt_2 &= \iint_{A_0}\left[w(t_1)+\beta\int v^0(t_1,\omega_2)d\omega_2\right]dt_1dt_2\\ &+\iint_{B_0}\left[n(t_2)+\beta\int v^0(\bar{t}_1,\omega_2)d\omega_2\right]dt_1dt_2\\ &+\iint_{C_0}\left[-c+\beta\int v^0(\omega_1,\omega_2)d\omega_1 d\omega_2\right]dt_1dt_2\end{aligned}$$

where A_0, B_0, C_0 are the (t_1,t_2) values for which job, OLF, search are best. (A_0 is defined above. [sic]) Since all of $w(\cdot)$, $n(\cdot)$, $v^0(\cdot,\cdot)$ are integrable, each term $<\infty$, and $(Tv^0)(t_1,t_2)$ is integrable.

A good starting value for $v^0(t_1,t_2)$ is $v^0(t_1,t_2)\equiv 0$, which satisfies (A.4). By Brauer's fixed point theorem,

$$v^*(t_1,t_2) = \lim_{i\to\infty}\left(T^i v^0\right)(t_1,t_2)$$

will satisfy (A.4), and will have $v^*(t_1,t_2)<\infty$ for $w(t_1)<\infty$, $n(t_2)<\infty$. QED

II. B. Three State Non-Stationary Model

The equation of interest is (9), reproduced as equation (A.15)

(A.15) $V(w,n,z) = \max\{[w+\beta EV(w,N,Z)],[n+\beta EV(0,N,Z)],[-c+\beta EV(W,N,Z)]\}$

where

$$EV(w,N,Z) = \int_\Omega V(w,N(n,z,\omega),Z(z,\omega))dP(\omega)$$

$$EV(W,N,Z) = \int_\Omega V(W(w,z,\omega),N(n,z,\omega),Z(z,\omega))dP(\omega) \quad .$$

We want to find if the functional fixed point equation,

(A.16) $$v(w,n,z) = (Tv)(w,n,z)$$

where T is the operation on the right hand side of (A.15), has a unique fixed point, $v^*(w,n,z)$. Our claim is that

> Equation (A.16) (or equation (A.15)) has a unique solution within the class of continuous functions $v : [0,M] \times [0,M] \times [0,M] \rightarrow \Re$.

Proof:
We assume that the random function W, N, Z are all bounded by 0 below and $M<\infty$ above. From the definition of T, it is clear that if we start with a continuous function, the continuity will be preserved. (The maximum of continuous functions is also continuous.) The main item we must prove to show that Brauer's fixed point theorem holds is

(A.17) $$d(Tv^0,Tv^1) \le \beta\, d(v^0,v^1) \quad \text{for } 0<\beta<1\text{, any } v^0, v^1$$

$$d\left(Tv^0, Tv^1\right) = \sup_{\substack{0 \le w \le M \\ 0 \le n \le M \\ 0 \le z \le M}} \left| \left(Tv^0\right)(w,n,z) - \left(Tv^1\right)(w,n,z) \right|$$

The proof of (A.17) follows much the same lines as proving (A.5) above. We define sets A_0 , B_0 , C_0 , A_1 , B_1 , and C_1 analogously to those used above. For example,

$$A_0 = \left\{(w,n,z) : w + \beta E v^0(w,N,Z) \le \max\left\{\left[n + \beta E v^0(0,N,Z)\right], \left[-c + \beta E v^0(W,N,Z)\right]\right\}\right\}$$

Then $d(Tv^0, Tv^1)$ can be written in the same form as (9) [sic] above. For a fixed z , say z^* , the arguments leading from (7) to (14a) [sic] would hold, since z would not vary. Thus we immediately have

$$\sup_{\substack{0 \le w \le M \\ 0 \le n \le M}} \left| \left(Tv^0\right)(w,n,z^*) - \left(Tv^1\right)(w,n,z^*) \right| \le \sup_{\substack{0 \le w \le M \\ 0 \le n \le M}} \beta \left| v^0(w,n,z^*) - v^1(w,n,z^*) \right|$$

for every z^* . Thus

$$\sup_{\substack{0 \le w \le M \\ 0 \le n \le M \\ 0 \le z \le M}} \left| \left(Tv^0\right)(w,n,z) - \left(Tv^1\right)(w,n,z) \right| \le \sup_{\substack{0 \le w \le M \\ 0 \le n \le M \\ 0 \le z \le M}} \beta \left| v^0(w,n,z) - v^1(w,n,z) \right|$$

In other words

$$d(Tv^0,Tv^1) \le \beta\, d(v^0,v^1) \ .$$

This is what we had to prove to prove Brauer's fixed point theorem. In other words, (A.15) or (A.17) have a unique solution. QED

II. C. Three State Model – Continuous Time

Rather than try to prove that the continuous time value function exists and is unique, we will show that the continuous time operator is equivalent to the discrete time operator. This result is interesting in itself for it implies that the discrete and continuous problems are not very different, but it also allows us to use the uniqueness and existence proofs from the discrete time case.

We start with the stationary model with involuntary layoff, i.e., the model presented in Section II. C:

(A.18)

$$V(w,n) = \frac{1}{\eta + r}\max\left\{\begin{array}{l}\left[w + \mu EV(w,N) + \lambda_1 EV\left(\max(W,w),n\right)\right) + (\lambda_2 + \lambda_3)V(w,n) + \delta V(0,n)\right], \\ \left[n + \mu EV(0,N) + \lambda_2 EV(W,n) + (\lambda_1 + \lambda_3 + \delta)V(0,n)\right], \\ \left[-c + \mu EV(0,N) + \lambda_3 EV(W,n) + (\lambda_1 + \lambda_2 + \delta)V(0,n)\right]\end{array}\right\}$$

where $\eta \equiv \mu+\lambda_1+\lambda_2+\lambda_3+\delta$. If we need [sic] the following definitions of vector-valued random variables

(A.19)

$$X_1 = \begin{cases}(w,N) & \text{with probability } \mu/\eta \\ \left(\max(W,w),N\right) & \text{with probability } \lambda_1/\eta \\ (w,n) & \text{with probability } (\lambda_2+\lambda_3)/\eta \\ (0,n) & \text{with probability } \delta/\eta\end{cases}$$

$$X_2 = \begin{cases}(0,N) & \text{with probability } \mu/\eta \\ (W,n) & \text{with probability } \lambda_2/\eta \\ (0,n) & \text{with probability } (\lambda_1+\lambda_3+\delta)/\eta\end{cases}$$

$$X_3 = \begin{cases}(0,N) & \text{with probability } \mu/\eta \\ (W,n) & \text{with probability } \lambda_3/\eta \\ (0,n) & \text{with probability } (\lambda_1+\lambda_2+\delta)/\eta\end{cases}$$

Also define $\beta = 1/(1+r/\eta)$ and note that

$$\frac{1}{\eta + r} = \frac{1}{\eta}\frac{1}{1+\frac{r}{\eta}} = \beta\frac{1}{\eta}$$

Then (A.18) can be written as

(A.20) $$V(w,n) = \max\left\{\left[\frac{1}{\eta}\beta w + \beta EV(X_1)\right], \left[\frac{1}{\eta}\beta n + \beta EV(X_2)\right], \left[-\frac{1}{\eta}\beta c + \beta EV(X_3)\right]\right\}$$

With some minor modifications this is the same equation for the discrete model, equation (1). To see the equivalence we must note:

1) η depends on the time unit chosen. We could choose our time unit so that $\eta=1$.
2) In deriving (1) we assumed wages were paid at the beginning of a period. If it were paid at the end, w , n , -c would all have been multiplied by β .
3) Equation (1) uses the definitions of X_1 , X_2 , X_3 as

$$X_1 = (w,N)$$

$$X_2 = (0,N)$$

$$X_3 = (W,N)$$

We could have introduced more complex wage offer structures and the proofs of existence and uniqueness of (1) would still have stood. The proof of existence and uniqueness of (1) in Appendix II.A above does not depend on the nature of the random variables X_1 , X_2 , X_3 . The definitions (A.20) above are perfectly legitimate because they are restricted to the region $[0,M] \times [0,M]$.

For the non-stationary continuous-time model the proof follows the same strategy as for the discrete time model. We show that, for every z^*

$$\sup_{\substack{0 \le w \le M \\ 0 \le n \le M}} \left| \left(Tv^0\right)(w,n,z^*) - \left(Tv^1\right)(w,n,z^*) \right| \le \sup_{\substack{0 \le w \le M \\ 0 \le n \le M}} \beta \left| v^0(w,n,z^*) - v^1(w,n,z^*) \right|$$

Then immediately

$$d(Tv^0,Tv^1) \le \beta \, d(v^0,v^1) \ .$$

BIBLIOGRAPHY

Blackwell, D. (1965), "Discounted Dynamic Programming," *Annals of Mathematical Statistics*, 36: 226-235 [NB – not in original]

Boyce, W.E. and DiPrima, R.C., (1969), *Elementary Differential Equations and Boundary Value Problems*, 2nd ed., New York, Wiley.

Braun, M., (1978), *Differential Equations and their Applications*, 2nd ed., Springer-Verlag, New York.

Lippman, S.A. and McCall, J.J., (1976a) "Job Search in a Dynamic Economy," *Journal of Economic Theory*, v. 12, No. 3, June (1976).

Lippman, S.A. and McCall, J.J., (1976b) "The Economics of Job Search," *Economic Inquiry*, v. 14 Part I (June 1976); Part II (Sept. 1976).

Wouk, A., (1979), *A Course of Applied Functional Analysis*, John Wiley and Sons, New York.